Clusters and Recurrence in the Two-Dimensional Zero-Temperature Stochastic Ising Model

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Abstract

We analyze clustering and (local) recurrence of a standard Markov process model of spatial domain coarsening. The continuous time process, whose state space consists of assignments of $+1$ or -1 to each site in $\mathbf{Z}^2$, is the zero-temperature limit of the stochastic homogeneous Ising ferromagnet (with Glauber dynamics): the initial state is chosen uniformly at random and then each site, at rate one, polls its 4 neighbors and makes sure it agrees with the majority, or tosses a fair coin in case of a tie. Among the main results (almost sure, with respect to both the process and initial state) are: clusters (maximal domains of constant sign) are finite for times $t < \infty$, but the cluster of a fixed site diverges (in diameter) as $t \to \infty$; each of the two constant states is (positive) recurrent. We also present other results and conjectures concerning positive and null recurrence and the role of absorbing states.

Mathematics Subject Classification 2000: 60K35, 82C22, 82C20, 60J25.

Key words and phrases: Clusters, Recurrence, Percolation, Stochastic Ising Model, Transience, Zero-Temperature, Coarsening.

Abbreviated title: Recurrence in the Stochastic Ising Model.

1 Synopsis

Consider the following Markov process, whose state σ^t at (continuous) time t is an assignment to each site in $\mathbf{Z}^2$ of $+1$ or -1. The initial state is chosen uniformly at random

and then with rate one each site changes its value (resp., determines its value by a fair coin toss) if it disagrees with three or four (resp., exactly two) of its four nearest neighbors. This process has been much studied in the physics literature as a model of "domain coarsening" (see, e.g., [2]): clusters of constant sign (either $+1$ or -1) shrink or grow or split or coalesce as their boundaries evolve. A more detailed definition along with some physical motivation will be given in Section 2 below.

One focus of this paper is the study of the asymptotic growth of clusters. Let $R_*(t)$ (resp., $R^*(t)$) denote the Euclidean distance from the origin to the closest (resp., farthest) site in its cluster that is next to the cluster boundary (i.e., that has a neighbor of opposite sign). It was previously proved [17] that almost surely, each site flips (i.e., changes its value) infinitely often and thus $\liminf_{t\to\infty} R_*(t) = 0$. Among our main results are the following [with references in brackets to later in the paper]:

- For any t, almost surely, $R^*(t) < \infty$; i.e., there is no percolation at time t [Proposition 3.2].
- $R^*(t) \to \infty$ almost surely [Proposition 3.1].
- $\limsup_{t\to\infty} R_*(t) = \infty$ almost surely [Corollary 5.1].

The last of the three results just mentioned is a corollary of our other main focus — the analysis of (local) recurrence of states σ or (measurable) subsets $\mathcal{M}$ of states. We say that σ is *recurrent* (for a given ω in the underlying probability space) if $\sigma^{t_k} \to \sigma$ along some subsequence $t_k \to \infty$. We say that $\mathcal{M}$ is recurrent if some $\sigma \in \mathcal{M}$ is recurrent. (Related notions of recurrence for interacting particle systems are studied in a recent paper by Cox and Klenke [4].) A non-recurrent σ or $\mathcal{M}$ will naturally be called *transient*. Although it has not yet been proved that the probability distribution μ^t of σ^t has a unique limit as $t \to \infty$, nevertheless, we will classify a recurrent σ (or $\mathcal{M}$) as *positive recurrent* if some subsequence limit μ of μ^t (these always exist by compactness) has $\mu(\{\sigma\}) > 0$ (or $\mu(\mathcal{M}) > 0$); otherwise it is classified as *null recurrent*.

The formulation of our recurrence results involves the absorbing states of the process, i.e., those states in which every site agrees with at least three of its neighbors. Note that these need not be recurrent, since our definition of recurrence is with respect to a uniformly random initial state. It is easy to see that besides the two constant states (identically $+1$ or identically -1), the absorbing states are those whose clusters are all either half spaces or infinite strips, and their cluster boundaries are all doubly-infinite flat lines (either all vertical or all horizontal) separated from each other by distance at least two. We prove the following, where μ denotes *any* subsequence limit of μ^t:

- $\mu(\{\text{non-absorbing states}\}) = 0$ [Theorem 2] or equivalently, the rate of flips at the origin tends to zero, in probability. Thus, almost surely, the set of non-absorbing states is not positive recurrent.
- Almost surely, each of the two constant states is positive recurrent [Theorems 3, 4].

A natural conjecture (see Remark 5.1) is that $\mu(\{\text{non-constant states}\}) = 0$ (or equivalently, that the density of cluster boundaries tends to zero, or equivalently that $R_*(t) \to \infty$ in probability) and thus that, almost surely, the set of non-constant states is not positive recurrent.

We also prove:

- Almost surely, the set of non-constant absorbing states is recurrent [Theorem 4].
- Almost surely, the set of non-absorbing states is null recurrent [Remark 5.2].

The non-absorbing states of the last-mentioned result may be restricted to those with a cluster boundary that is flat except for a single step of unit size next to the origin (providing the origin with a tie among its four neighbors). There are two plausible scenarios concerning the exact class of null recurrent states. To explain these, we first note that cluster boundaries in recurrent states must be doubly infinite and *monotonic*, i.e., with every finite segment either flat or else having a Southwest–Northeast (resp., Northwest–Southeast) orientation [Lemma 4.2]. If there were more than a single such "domain wall" in a recurrent state, there would be further restrictions concerning their relative locations (see, e.g., the proof of Lemma 4.3). But we conjecture that this does not occur; indeed, we expect that one of the following two possibilities occurs.

- Scenario 1: Almost surely, the null recurrent states are exactly all states with single infinite monotonic domain walls while the set of all other nonconstant states is transient.
- Scenario 2: Almost surely, the null recurrent states are exactly all states with single infinite domain walls that are either completely flat or else have a single step, with unit size, while the set of all other nonconstant states is transient.

2 Introduction

The behavior of different kinds of magnetic systems following a deep quench is a central topic in the study of their nonequilibrium dynamics. Physically, a deep quench is when a system that has reached equilibrium at some high temperature T_1 has its temperature rapidly reduced to a much lower T_2. In this paper, as in much of the theoretical physics literature, we take $T_1 = \infty$ and $T_2 = 0$. Rigorous and nonrigorous results have been obtained on different questions that arise naturally in this context, such as the formation of domains, their subsequent evolution, spatial and temporal scaling properties and related problems (for a review, see [2]). In particular, in the context of zero-temperature, stochastic Ising models with nearest-neighbor interactions, the question of whether the spin configuration eventually settles down to a final state has been addressed rigorously and answered for a number of different models [12, 17]. A closely related issue is that of persistence, concerning the fraction of sites that have not flipped at all by time t, and its asymptotic behavior as $t \to \infty$ [5, 6, 18, 21].

In this paper, we consider the zero-temperature, stochastic (homogeneous) Ising model σ^t on $\mathbf{Z}^d$ with nearest-neighbor ferromagnetic interactions. In dimension one, the model is the same as the $d = 1$ voter model, which is well understood (see, e.g., Chapter 2 of [7] or Chapter V of [16] and references therein). We study in detail the case $d = 2$, for which it is known that there is not a unique limiting state [17], and consider questions about the limits along subsequences of time and the nature of clusters of sites of the same sign. The aim of the paper is to give a picture of the system for very long times, showing what kinds of events and states are seen as $t \to \infty$ and what are instead "forbidden". These are basically questions of recurrence and transience.

In the remainder of this section, we define the model precisely and discuss some results and open problems about this and related models. The stochastic process $\sigma^t = \sigma^t(\omega)$ corresponds to the zero-temperature limit of Glauber dynamics for an Ising model with (formal) Hamiltonian

$$\mathcal{H} = - \sum_{\substack{\{x,y\} \\ ||x-y||=1}} J_{x,y} \sigma_x \sigma_y. \tag{1}$$

Here the state space is $\mathcal{S} = \{-1, +1\}^{\mathbf{Z}^d}$, the space of (infinite-volume) spin configurations σ, and $\| \cdot \|$ denotes Euclidean length. The initial spin configuration σ^0 is chosen from a symmetric Bernoulli product measure (denoted μ^0), corresponding physically to a deep quench from infinite temperature. (We note that the case of an *asymmetric* initial μ^0 has also been studied, both on $\mathbf{Z}^d$ [10] and on other lattices [15].) The continuous time dynamics is defined by means of independent (rate 1) Poisson processes at each site x corresponding to those times (which we think of as "clock rings") when a spin flip ($\sigma_x^{t+0} = -\sigma_x^{t-0}$) is *considered*. If the change in energy,

$$\Delta \mathcal{H}_x(\sigma) = 2 \sum_{y: \|x-y\|=1} J_{x,y} \sigma_x \sigma_y, \tag{2}$$

is negative (or zero or positive), then the flip is done with probability 1 (or 1/2 or 0). We think of associating a fair coin toss to each clock ring, which we use as a tie-breaker only when $\Delta \mathcal{H}_x(\sigma) = 0$. Let us denote by P_{dyn} the probability measure for the dynamics realizations of clock rings and tie-breaking coin tosses and then denote by $P = \mu^0 \times P_{dyn}$ the joint probability measure on the space Ω of the initial configurations σ^0 and realizations of the dynamics. An element of Ω will be denoted by ω.

The model that we will study in this paper is the homogeneous ferromagnet, where $J_{x,y} \equiv 1$ for all $\{x, y\}$. In this model, when the clock at site x rings, σ_x flips with probability 1/2 if it disagrees with exactly d neighbors and with probability 1 if it disagrees with more than d neighbors; it does not flip if it disagrees with less than d neighbors. In the first case, the spin flip leaves the energy unchanged, $\Delta \mathcal{H}_x(\sigma) = 0$, while in the second case the spin flip lowers the energy, $\Delta \mathcal{H}_x(\sigma) < 0$. A very useful result of Nanda, Newman and Stein (see Theorem 3 and the following remark in [17]) states that the number of energy lowering spin flips at any site is almost surely finite.

Disordered models, in which a realization $\mathcal{J}$ of the $J_{x,y}$'s is chosen from the (independent) product measure of some probability measure ν, are also studied in the literature (see, e.g., [17, 20]) and have different properties, but we will not deal with those models here. For the homogeneous ferromagnet in dimensions $d = 1$ [1, 3] and $d = 2$ [17], $\sigma^\infty(\omega) = \lim_{t\to\infty} \sigma^t(\omega)$ does not exist; indeed, for almost every ω and for every x, $\sigma_x^t(\omega)$ flips infinitely many times. For $d > 2$, little is known rigorously, but numerical studies [21] suggest that the same is true up to $d = 4$, while $\sigma^\infty(\omega)$ might perhaps exist for $d > 4$.

When the limit does not exist, it is natural to ask what happens to the measure describing the state of the system as $t \to \infty$. A natural way to approach this question is by looking at the clusters of sites with the same sign and at the domain walls between such clusters. The two descriptions are basically equivalent and we will use both, depending on the type of problem.

It is a direct consequence of a result of Harris [13, 16] that the distribution μ^t of σ^t satisfies the FKG property for any time t. In dimension $d = 2$, we will use this and a result of Gandolfi, Keane and Russo [11] to show that at any time t, neither $+1$ nor -1 spins percolate; i.e., the clusters are almost surely finite. We will also show, however, that the diameter of the cluster at the origin almost surely diverges as $t \to \infty$, for any d. In dimension two, it is a natural conjecture that for large enough times the system will be, with P-probability close to 1, locally in a $+1$ or in a -1 phase or, equivalently, that the density of domain walls tends to zero. In fact, it is expected that the density is of order $t^{-1/2}$ as $t \to \infty$ (see, e.g., [2]). Although we are not able to prove this, Theorem 3 in Section 5 below points in that direction.

If the above conjecture is true, then, by symmetry, it automatically follows that the distribution μ^t of σ^t has the unique limit, as $t \to \infty$, of $\frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$, where δ_η is the probability measure assigning probability one to the constant ($\equiv \eta$) spin configuration. But μ^t is the overall distribution of σ^t, taking into account that the initial state is random and distributed by the Bernoulli product measure μ^0. If instead, we condition on σ^0 and consider the conditional distribution $\mu^t[\sigma^0]$ (for almost every σ^0), it is unclear whether that should still converge as $t \to \infty$ to $\frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$, or rather there should be multiple subsequence limits (presumably all of the form $\alpha\delta_{+1} + (1-\alpha)\delta_{-1}$) along different σ^0-dependent subsequences of time. The latter situation would be an example of "Chaotic Time Dependence" (CTD) [8] (see also [19]). CTD is known not to occur for the $d = 1$ version of our model (equivalent to the voter model), but has been proved to occur in a disordered $d = 1$ voter model [8, 9].

3 Percolation results

In this section we present two propositions about clusters of constant sign. For every $x \in \mathbf{Z}^d$, let us denote by $C_x(t) = C_x(\sigma^t)$ the *cluster at site x at time t*. $C_x(t)$ is defined as the maximal subset of $\mathbf{Z}^d$ satisfying the following properties:

- $x \in C_x(t)$,

- $C_x(t)$ is connected (in the sense that if y and z are both in $C_x(t)$, there exists a sequence ζ_i, $i = 0, 1, 2, \ldots, n$, of sites of $C_x(t)$ with $||\zeta_{i+1} - \zeta_i|| = 1$ and with $\zeta_0 = y$ and $\zeta_n = z$),
- if $y, z \in C_x(t)$, then $\sigma_y^t = \sigma_z^t$.

$|C_x(t)|$ will denote the number of sites in $C_x(t)$. The origin is denoted by o and so $C_o(t)$ is the cluster at the origin.

Our first result, and the only one valid for all d, concerns the growth of $C_o(t)$ with time. It remains valid in a very general setting (see Theorem 3 and the following remark in [17]) and in particular applies to our Markov process when the initial state is chosen according to any translation-invariant measure.

Proposition 3.1. *For any d, the size of the cluster at the origin diverges almost surely as $t \to \infty$: $\lim_{t\to\infty} |C_o(t)| = \infty$.*

Proof. We will prove the proposition by contradiction. Suppose that the conclusion is not true; then with positive probability, $\liminf_{t\to\infty} |C_o(t)| < \infty$ and so there exist $M < \infty$ and a sequence of times $\{t_k\}_{k\in\mathbf{N}}$ with $t_k \to \infty$ such that $|C_o(t_k)| < M$ for $k = 1, 2, \ldots$. Without loss of generality, we may assume that $t_{k+1} > t_k + 1$. There are only finitely many shapes (lattice animals) that the cluster at the origin can have at times t_k when $|C_o(t_k)| < M$. For each such lattice animal, there is some ordered finite sequence of clock rings and outcomes of tie-breaking coin tosses inside a fixed finite ball that would cause the cluster to shrink to a single site at the origin which would then have an energy lowering spin flip. It follows that for some $\delta > 0$ and any $\sigma \in \mathcal{S}$ such that $|C_o(\sigma)| < M$,

$$P(\text{ origin flips at time } t \in (t_k, t_k + 1) \text{ with } \Delta\mathcal{H}_o < 0 \mid \sigma^{t_k} = \sigma) \geq \delta. \tag{3}$$

By the Markov property of the process and our supposition that the conclusion of the proposition is false, this would imply that the spin at the origin σ_o flips infinitely many times with $\Delta\mathcal{H}_o < 0$ with positive probability, which contradicts a result of Nanda, Newman and Stein [17] mentioned in Section 2 of the paper. □

We set $d = 2$ now and for the rest of the paper. Our second result is a direct consequence of a result of Harris [13, 16] and one of Gandolfi, Keane and Russo [11].

Proposition 3.2. *At any (deterministic) time, there is no percolation of clusters of spins of the same sign: $P(|C_o(t)| = \infty) = 0$ for all $t \geq 0$.*

Proof. First note that the measure μ^t describing the state σ^t of the system at time t is invariant and ergodic under $\mathbf{Z}^2$-translations. This is so because the same is true for both μ^0 and P_{dyn} and hence also for P. Applying a result of Harris [13, 16], we also have that μ^t satisfies the FKG property, i.e., increasing functions of the spin variables are positively correlated (this follows from the FKG property of μ^0 and the attractivity of the Markov process). Then it follows from a result of Gandolfi, Keane and Russo [11] that

if percolation of, say, $+1$ sites were to occur, all the -1 clusters would have to be finite. Because of the symmetry of the model under a global spin flip, however, percolation of $+1$ sites with positive probability implies the same for -1 sites. Then, using the ergodicity of the measure, we would see simultaneous percolation of both signs, thus obtaining a contradiction. $\square$

4 Preliminary recurrence results

We now introduce the contour representation in the dual lattice $\mathbf{Z}^{2^*} \equiv \mathbf{Z}^2 + (1/2, 1/2)$, following the notation of [12]. A (dual) site in $\mathbf{Z}^{2^*}$ may be identified with the plaquette p in $\mathbf{Z}^2$ of which it is the center. The edge $\{x, y\}^*$ of $\mathbf{Z}^{2^*}$, dual to (i.e., perpendicular bisector of) the edge $\{x, y\}$ of $\mathbf{Z}^2$, is said to be *unsatisfied* (with respect to a given spin configuration $\sigma \in \mathcal{S}$) if $\sigma_x \neq \sigma_y$ (and *satisfied* otherwise). Denote by Γ the set of unsatisfied (dual) edges. Given a finite rectangle Λ of $\mathbf{Z}^2$, $\Lambda^* \subset \mathbf{Z}^{2^*}$ consists of the dual sites corresponding to the plaquettes contained in Λ. $\Gamma(\Lambda^*)$ is the set of unsatisfied (dual) edges bisecting the edges connecting sites in Λ. Note that the outermost edges of $\Gamma(\Lambda^*)$ have one endpoint just outside of Λ^*. A (site self-avoiding) path in $\mathbf{Z}^{2^*}$ using only unsatisfied edges will be called a *domain wall*; it is simply a path along the cluster boundaries of σ. If $\Gamma(\Lambda^*)$ is not empty, it can contain one or more domain walls. Since domain walls are the boundaries between clusters of sites with different sign, they can always be extended to form a closed loop or a doubly infinite path. Every $\Gamma(\mathbf{Z}^{2^*})$ configuration corresponds to two spin configurations related by a global spin flip. The Markov process σ^t determines a process Γ^t, that is easily seen to also be Markovian. The transition associated with a spin flip at $x \in \mathbf{Z}^2$ is a local "deformation" of the contour Γ^t at the (dual) plaquette that contains x; this deformation interchanges the satisfied and unsatisfied edges of that plaquette. The only transitions with nonzero rates are those where the number of unsatisfied edges starts at $k = 4$ or 3 or 2 and ends at 0 or 1 or 2, respectively; transitions with $k = 4$ or 3 (resp. 2) correspond to energy-lowering (resp., zero-energy) flips and have rate 1 (resp. 1/2). We will continue to use the terms flip, energy-lowering, etc. for the transitions of Γ^t.

We continue with some definitions and lemmas.

Definition 4.1. *Let Z_t be a continuous-time Markov process with state space $\mathcal{Z}$ and time homogeneous transition probabilities. For A a (measurable) subset of $\mathcal{Z}$ we say that A* recurs *if $\{\tau : Z_\tau \in A\}$ is unbounded, and we say that A is* eventually absent (e-absent) *if it recurs with zero probability. For our stochastic Ising model, the restriction $\sigma|_\Lambda$ of some $\sigma \in \mathcal{S}$ to $\Lambda \subset \mathbf{Z}^2$ will be called e-absent if $\{\sigma' \in \mathcal{S} : \sigma'|_\Lambda = \sigma|_\Lambda\}$ is e-absent.*

Note that if a contour event A, specified by $\Gamma(\Lambda^*)$, is e-absent, then any $\sigma|_\Lambda$ consistent with A is also e-absent. Note further that our definition in Section 1 for recurrence of $\sigma \in \mathcal{S}$ is that for every finite Λ, the restriction $\sigma|_\Lambda$ recurs. Thus almost sure transience is implied by, but not equivalent to e-absence of $\sigma|_\Lambda$ for all large Λ since, a priori, it could be that $\sigma|_\Lambda$ recurs with nonzero probability, tending to zero as $\Lambda \to \mathbf{Z}^2$.

By Q_L we denote the square of size $2L+1$ centered at the origin, that is the set of all $x=(x_1,x_2)\in \mathbf{Z}^2$ such that $x_i \in \{-L,\dots,L\}$. ($Q_L(x)$ will be used later to denote the square of size $2L+1$ centered at x; i.e., $Q_L(x)=Q_L+x$).

Lemma 4.1. *If A is e-absent, then $P(Z_t \in A) \to 0$ as $t\to\infty$. In particular, if we denote by S_L the set of $\sigma \in \mathcal{S}$ such that $\sigma|_{Q_L}$ is e-absent, then*

$$\lim_{t\to\infty} P(\sigma^t \in S_L) = 0. \tag{4}$$

Proof. Suppose the lemma were false. Then there would exist $\delta>0$ and a sequence of times $t_k \uparrow \infty$ such that for all k, $P(Z_{t_k}\in A)>\delta$. But then it would follow that

$$P(Z_t \text{ recurs }) > \delta, \tag{5}$$

contradicting the e-absence of A, as a consequence of the standard fact that

$$P(B_k) > \varepsilon \text{ for all } k\in\mathbf{N} \text{ implies } P(B_k \text{ occurs infinitely often }) > \varepsilon. \; \square \tag{6}$$

The following lemma is essentially the same as Lemma 8 of [12] (where a more detailed proof may be found). We say that a domain wall in $\mathbf{Z}^{2^*}$ is *monotonic* if, for one of the two directed path versions of the domain wall, either every move is to the North or East or else every move is to the South or East.

Lemma 4.2. *The event $\{\Gamma^t(Q_L^*)$ contains a non-monotonic domain wall$\}$ is e-absent.*

Proof. A non-monotonic domain wall in $\Gamma(Q_L^*)$ can always be modified through local deformations (corresponding to appropriate spin flips of sites in Q_L) to give a contour configuration $\Gamma'(Q_L^*)$ with three (or four) domain wall edges surrounding some plaquette of Q_L^*. The corresponding spin configurations then have a site that disagrees with three (or four) neighbors, which can undergo an energy-lowering spin flip. As in the proof of Proposition 3.1, the existence of such local deformations (or sequence of spin flips) means that there is a bounded away from zero probability (corresponding to an appropriate sequence of clock rings and tie-breaking coin tosses) of an energy-lowering spin flip during the next unit time interval. If the claim were not true, then the event that $\Gamma(Q_L^*)$ contains a non-monotonic domain wall would recur and there would be a nonzero probability of infinitely many energy lowering spin flips in Q_L, which would contradict an already mentioned theorem of Nanda, Newman and Stein [17]. $\square$

The next lemma provides a geometric upper bound which is one of the key technical results of this paper. In particular, it will be used in the proof of Theorem 1 below. Before we can state the lemma, we need a definition. For a given $\sigma\in\mathcal{S}$, let $M_L(\sigma)$ denote the total number of corners in $\Gamma(Q_L^*)$, i.e., pairs of perpendicular edges that meet at a site in Q_L^*. S_L^c, the complement of S_L, is the set of σ's such that $\sigma|_{Q_L}$ is not e-absent.

Lemma 4.3. *For $\sigma\in S_L^c$, the number $M_L(\sigma)$ of corners is bounded by*

$$\max_{\sigma\in S_L^c} M_L(\sigma) \le 4(2L+1). \tag{7}$$

Proof. By Lemma 4.2, any $\Gamma(Q_L^*)$ that is not e-absent may be partitioned into edge-disjoint monotonic domain walls, $\gamma_1, \ldots, \gamma_m$, whose endpoints are just outside of Q_L^*. Let us denote by $M(\gamma_i)$ the number of corners in γ_i. Clearly, if $m = 1$, then $M_L = M(\gamma_1) \leq 2(2L+1)$. When $m > 1$, we need to consider the geometric constraints on the γ_i's required for $\sigma \in S_L^c$. First, there are the special $m = 2$ cases where $\Gamma(Q_L^*)$ is a "cross," i.e., the union of one flat horizontal and one flat vertical domain wall; here $M_L = 4 \leq 4(2L+1)$. We claim that in all remaining cases, the γ_i's are site-disjoint and in fact, their spanning rectangles $R(\gamma_i)$, defined so that two of the vertices of $R(\gamma_i)$ are the endpoints of γ_i, must also be site-disjoint.

Note that, depending on whether γ_i connects two opposite or two adjacent sides of the boundary of Q_L^*, the rectangle $R(\gamma_i)$ can be classified as either vertical or horizontal or as one of four corner-types. The reason the $R(\gamma_i)$'s must be site-disjoint (except in a cross configuration) is that otherwise there would exist a sequence of spin flips (corresponding to a sequence of clock-rings and tie-breaking coin tosses) in Q_L that would deform $\Gamma(Q_L^*)$ into a contour configuration with a non-monotonic domain wall (see Figure 1 and Lemmas 9 and 10(i) of [12]). As in the proof of Lemma 4.2, this shows that intersecting $R(\gamma_i)$'s must be e-absent.

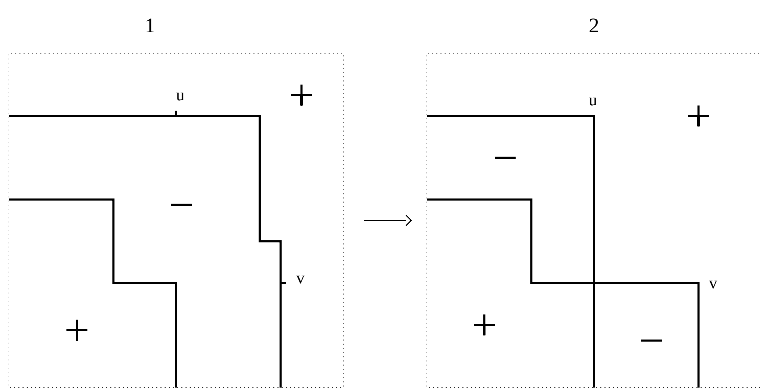

Figure 1: An example in which configuration 1 is deformed to configuration 2 by spin flips with $\Delta\mathcal{H} = 0$. Configuration 2 has non-monotonic domain walls.

When the γ_i's are site-disjoint, $M_L = \sum_i M(\gamma_i)$, and if $R(\gamma_i)$ has sides of length l_1^i and l_2^i, then $M(\gamma_i) \leq 2\min(l_1^i, l_2^i)$. Now it is easily seen that the sum of the shorter sides of these non-overlapping $R(\gamma_i)$'s is bounded by twice the linear dimension of Q_L (i.e. $2(2L+1)$), yielding $M_L \leq 4(2L+1)$ as claimed; indeed the worst case is when $m = 4$ and each γ_i is of corner-type, with the $R(\gamma_i)$'s almost overlapping. $\square$

One of the main results of the paper, Theorem 2 below, concerns the probability that, for large t, σ^t is locally in an absorbing state. As noted in Section 1, the absorbing states are constant either on infinite horizontal lines or infinite vertical lines. The next result of this section is a more technical theorem about the density of domain wall corners, from which Theorem 2 follows easily. For $t \geq 0$, and $x^* \in \mathbf{Z}^{2*}$, we define $F_{x^*}(t)$ to be the event

that there is a corner in $\Gamma^t(\mathbf{Z}^{2*})$ with vertex x^*.

Theorem 1. *For any* $x^* \in \mathbf{Z}^{2*}$, $\lim_{t\to\infty} P(F_{x^*}(t)) = 0$.

Proof. If $F_{x^*}(t)$ occurs, then there is at least one (and at most four) corners at x^*. Thus $\tilde{M}_L(\sigma^t)$, the number of x^*'s in Q_L^* such that $F_{x^*}(t)$ occurs, is bounded by the number $M_L(\sigma^t)$ of corners, and so by translation invariance and elementary arguments,

$$\begin{aligned}
P(F_{x^*}(t)) &= E(1_{F_{x^*}(t)}) = \frac{1}{(2L)^2} E(\tilde{M}_L(\sigma^t)) \\
&= \frac{1}{(2L)^2}\left[P(\sigma^t \in S_L)E(\tilde{M}_L(\sigma^t)|\sigma^t \in S_L) + P(\sigma^t \in S_L^c)E(\tilde{M}_L(\sigma^t)|\sigma^t \in S_L^c)\right] \\
&\le \frac{1}{(2L)^2}\left[P(\sigma^t \in S_L)\cdot(2L)^2 + E(M_L(\sigma^t)\,|\,\sigma^t \in S_L^c)\right] \\
&\le P(\sigma^t \in S_L) + \frac{1}{(2L)^2}\max_{\sigma\in S_L^c} M_L(\sigma)\,. \qquad (8)
\end{aligned}$$

The proof is completed by using Lemmas 4.1 and 4.3 to observe that the two terms in the final expression can be made small by appropriate choice of large L and t. □

5 Main recurrence results

Let us now define two sets of states that play an important role in recurrence:

$$\mathcal{C} = \{\sigma \in \mathcal{S} \,:\, \sigma \equiv 1 \text{ or } \sigma \equiv -1\}, \qquad (9)$$

the set of constant spin configurations, and

$$\mathcal{A} = \{\text{ absorbing states }\}, \qquad (10)$$

consisting of the two constant spin configurations in $\mathcal{C}$ together with all spin configurations corresponding to contour configurations that are unions of doubly infinite flat domain walls (either all horizontal or all vertical) separated by at least two lattice spacings. These are the only absorbing states in $d = 2$, since only for these does every spin agree with a strict majority of its neighbors.

We also introduce the set of all spin configurations $\tilde{\sigma}$ that agree with a given subset of $\mathcal{S}$ inside the square Q_L: for $\mathcal{U} \subset \mathcal{S}$, define

$$\mathcal{U}_L = \{\tilde{\sigma} \in \mathcal{S} : \exists\, \sigma \in \mathcal{U} \text{ with } \sigma|_{Q_L} = \tilde{\sigma}|_{Q_L}\}. \qquad (11)$$

The next theorem states that σ^t, in any finite region, for large enough t, agrees with some absorbing state with probability arbitrarily close to 1. It is straightforward to see that this is equivalent to saying that the rate of flips at the origin goes to zero in probability as $t \to \infty$.

Theorem 2. *For all $L \in \mathbf{N}$, $\lim_{t\to\infty} P(\sigma^t \in \mathcal{A}_L) = 1$.*

Proof. We will prove the theorem by contradiction. If the claim is not true, there exist $L > 0$, $\delta > 0$ and a sequence t_k with $t_k \to \infty$ such that for all k, $P(\sigma^{t_k} \notin \mathcal{A}_L) > \delta$. Now, the event $\{\sigma^t \notin \mathcal{A}_L\}$ corresponds to either having

- at least one non-flat domain wall inside Q_L, or else
- two flat domain walls at distance one apart.

The first case corresponds to having at least one site $x^* \in Q_L^*$ for which the event $F_{x^*}(t)$, that there is a corner at x^*, occurs. The second case corresponds to a configuration that can become non-monotonic with one flip and is therefore e-absent. Thus

$$P(\sigma^t \notin \mathcal{A}_L) \leq \sum_{x^* \in Q_L^*} P(F_{x^*}(t)) + P(\sigma^t \in S_L), \tag{12}$$

and the right-hand side goes to zero as $t \to \infty$ by Theorem 1 and Lemma 4.1. This completes the proof. $\square$

The next theorem, combined with the first part of Theorem 4, shows that each of the two constant states is positive recurrent in the sense of Section 1.

Theorem 3. *Let $C_L^+(t)$ (resp., $C_L^-(t)$) denote the event that σ^t is constant and equal to $+1$ (resp., to -1) on the square Q_L. Then for any $L < \infty$,*

$$\liminf_{t\to\infty} P(C_L^+(t)) \; = \; \liminf_{t\to\infty} P(C_L^-(t)) \; \geq \; 1/4. \tag{13}$$

Proof. We introduce the following events that, like $C_L^+(t)$, are increasing in the FKG sense:

$$\begin{aligned} V_L^+(t) \;&=\; \{\omega : \sigma^t|_{Q_L}(\omega) \text{ has a vertical line of } 2L+1 \text{ sites that are } +1\}, & (14)\\ H_L^+(t) \;&=\; \{\omega : \sigma^t|_{Q_L}(\omega) \text{ has a horizontal line of } 2L+1 \text{ sites that are } +1\}. & (15)\end{aligned}$$

Note that $C_L^+(t) \subset V_L^+(t), H_L^+(t)$. We also define the corresponding events with $+$ replaced by $-$. With these definitions, we have

$$V_L^+(t) \cap H_L^+(t) \subset C_L^+(t) \cup \{\sigma^t \notin \mathcal{A}_L\} \tag{16}$$

and therefore

$$P(V_L^+(t) \cap H_L^+(t)) \leq P(C_L^+(t)) + P(\sigma^t \notin \mathcal{A}_L). \tag{17}$$

Using the fact that $V_L^+(t)$ and $H_L^+(t)$ are increasing events and the FKG property of the distribution of σ^t (see the proof of Proposition 3.2), we get

$$P(C_L^+(t)) \geq P(V_L^+(t))P(H_L^+(t)) - P(\sigma^t \notin \mathcal{A}_L). \tag{18}$$

Because of the "striped" nature of the absorbing states,

$$P(\sigma^t \in \mathcal{A}_L) = P(\{\sigma^t \in \mathcal{A}_L\} \cap H_L^+(t)) + P(\{\sigma^t \in \mathcal{A}_L\} \cap V_L^-(t)). \tag{19}$$

By the symmetries of the model, the two terms in the right hand side of (19) are equal and therefore

$$P(\sigma^t \in \mathcal{A}_L) = 2P(\{\sigma^t \in \mathcal{A}_L\} \cap H_L^+(t)). \tag{20}$$

Thus

$$\begin{aligned} P(H_L^+(t)) &= P(\{\sigma^t \in \mathcal{A}_L\} \cap H_L^+(t)) + P(\{\sigma^t \in \mathcal{A}_L\}^c \cap H_L^+(t)) & (21)\\ &= \frac{1}{2}P(\sigma^t \in \mathcal{A}_L) + P(\{\sigma^t \in \mathcal{A}_L\}^c \cap H_L^+(t)). & (22) \end{aligned}$$

Applying Theorem 2 and symmetry, we obtain

$$\lim_{t\to\infty} P(V_L^+(t)) = \lim_{t\to\infty} P(H_L^+(t)) = 1/2. \tag{23}$$

Taking the lim inf of both sides of (18) and using Theorem 2 once more, we have

$$\liminf_{t\to\infty} P(C_L^+(t)) \geq \lim_{t\to\infty} P(V_L^+(t))P(H_L^+(t)) = 1/4. \; \square \tag{24}$$

Remark 5.1. *A natural conjecture is that the system is in a constant ($+1$ or -1) state with probability approaching 1 as $t \to \infty$, i.e., for all L,*

$$\lim_{t\to\infty} P(\sigma^t \in \mathcal{C}_L) = 1. \tag{25}$$

This is equivalent to the conjecture for our $d = 2$ process that clustering *occurs:*

$$P(\sigma_x^t \neq \sigma_y^t) \to 0 \text{ as } t \to \infty \text{ for any } x, y \in \mathbf{Z}^2. \tag{26}$$

In $d = 1$, our process is the same as the one-dimensional voter model, for which clustering is known to occur [14] (see also, e.g., [7, 16]).

The next result is a corollary of Theorem 3. Recall that $R_*(t)$ denotes the Euclidean distance from the origin to the closest site in its cluster that is next to the cluster boundary. More precisely, given a subset Λ of $\mathbf{Z}^2$ and $x \in \mathbf{Z}^2$, define the distance $d(\Lambda, x) = \inf_{y\in\Lambda} ||x-y||$. The inner boundary of Λ is $\partial\Lambda = \{x \in \Lambda : \exists y \notin \Lambda \text{ with } ||x-y|| = 1\}$. Then $R_*(t) = d(\partial C_o(t), o)$.

Corollary 5.1. $\limsup_{t\to\infty} R_*(t) = \infty$ *almost surely.*

Proof. Define the event

$$A_L = \{\limsup_{t\to\infty} R_*(t) \geq L\}. \tag{27}$$

By Theorem 3 and (6) (applied for fixed L to the events $\{\sigma^t \in \mathcal{C}_L\}$ for a sequence of times), we have $P(A_L) \geq 1/2$ for every L. Then, letting $L \to \infty$, we have

$$P(\limsup_{t\to\infty} R_*(t) \;=\; \infty) \;\geq\; 1/2. \tag{28}$$

It is easy to see that the event A_∞ in (28) occurs if and only if its translation, i.e., the event that $\limsup_{t\to\infty} d(\partial C_x(t), x) = \infty$, occurs. Thus A_∞ is translation-invariant and by the translation-ergodicity of P, (28) implies that $P(A_\infty) = 1$, as desired. $\square$

As mentioned before, in dimension two, the state σ^t does not have a unique limit as $t \to \infty$ [17]; thus we are interested in the limits along subsequences of t. Let us introduce the (ω-dependent) set of all limiting states,

$$\mathcal{W} = \mathcal{W}(\omega) = \{\tilde{\sigma} \in \mathcal{S} : \exists\, t_k \uparrow \infty \text{ so that } \sigma_x^{t_k} \to \tilde{\sigma}_x \;\forall x \in \mathbf{Z}^2\}. \tag{29}$$

The following theorem concerns such subsequence limits. The first statement of the theorem means that there exists an ω-dependent sequence $t'_k \uparrow \infty$ so that $\sigma_o^{t'_k} = (-1)^k$ and $C_o(t_k) \supset Q_k$.

Theorem 4. *$\mathcal{W} \supset \mathcal{C}$ almost surely. Moreover, $\mathcal{W}$ contains a non-constant absorbing state with a flat domain wall passing next to the origin, almost surely.*

Proof. The proof that the constant $+1$ (resp., -1) state is in $\mathcal{W}$ is essentially the same as the proof of Corollary 5.1, but with the event A_L replaced by the event that $C_L^+(t)$ (resp., $C_L^-(t)$) occurs for an unbounded set of t's.

To prove the second part of the theorem, we consider the square Q_L (with L even, for a reason to be seen later) and use the fact [17] that σ_o^t flips infinitely many times almost surely. We restrict attention to times t greater than T_L, the almost surely finite time after which $\Gamma^t(Q_L^*)$ is not e-absent and hence satisfies various geometric constraints, including those discussed in the proof of Lemma 4.3.

There will almost surely be a sequence $t_k \to \infty$, of times (either just before or just after the flips of the origin) when there is a (monotonic) domain wall γ, whose endpoints are just outside Q_L^*, passing next to the origin and containing the origin in its spanning rectangle $R(\gamma)$. There are two possibilities: (1) γ connects two opposite sides of the boundary of Q_L^*, or (2) two adjacent sides.

Let B_L^1 (resp., B_L^2) denote the event that (1) (resp., (2)) occurs for an unbounded set of t's. Then $P(B_L^1) + P(B_L^2) \geq 1$ and so $\liminf_{L\to\infty} P(B_L^i) > 0$ either for $i = 1$ (we call this case 1) or $i = 2$ (case 2) or both. The γ of (1) may be either horizontal or vertical, so we express B_L^1 as the (not necessarily disjoint) union of events $B_L^{1,h}$ and $B_L^{1,v}$ according to whether a horizontal or a vertical γ recurs. By symmetry, these two events have equal

probability, so that in case 1, there is a subsequence $L_j \to \infty$ such that $P(B^{1,h}_{L_j}) > \delta > 0$ for all j.

Consider a time $t > T_{L_j}$ when such a horizontal γ is present. By the monotonicity and other geometric restrictions on the domain walls that follow because $\Gamma^t(Q^*_L)$ is not e-absent, it follows that there is a sequence of spin flips (with a bounded away from zero probability of occurring in the next unit time interval) that will deform such a $\Gamma^t(Q^*_L)$ into one where there is a horizontal flat domain wall γ' just under the origin and further such that $\sigma^t \in \mathcal{A}_L$ (i.e., inside Q_L, σ^t agrees with an absorbing state). Thus the event $\tilde{A}^h_{L_j}$, that $\sigma^t \in \mathcal{A}_L$ with a horizontal flat domain wall just under the origin for an unbounded set of t's, has $P(\tilde{A}^h_{L_j}) \geq P(B^{1,h}_{L_j}) > \delta > 0$ for all j.

Proceeding as in the proof of Corollary 5.1 (but using ergodicity only with respect to translations in the first coordinate), we conclude that $P(\tilde{A}^h_L$ occurs for all $L) = 1$, which, together with standard compactness arguments, completes the proof for case 1.

Case 2 is similar, but with an extra ingredient. Here we express B^2_L as the union of $B^{2,NE}_L$, where γ connects the North and East sides of the boundary of Q^*_L, and the three other directional possibilities. For a Northeast γ, we use spin flips to deform $\Gamma^t(Q^*_L)$ to a Γ' containing a contour that is horizontal and flat from the origin to the Eastern side of Q^*_L, and such that inside the smaller square $Q_{L/2}(L/2, 0)$, centered at $(L/2, 0)$, Γ' agrees with an absorbing state. By translating $(L/2, 0)$ to the origin and using translation invariance, we have in case 2 for some subsequence L'_j that $P(\tilde{A}^h_{L'_j/2}) \geq P(B^{2,NE}_{L'_j}) > \delta' > 0$ for all j. The remainder of the proof is as in case 1. $\square$

Remark 5.2. *We note that $\mathcal{W}$ is almost surely strictly larger than $\mathcal{A}$. Otherwise, it could not be the case that almost surely every site flips infinitely many times. Indeed, by arguments similar to those used for Theorem 4, there almost surely must be recurrent states that have a domain wall passing by the origin that is flat except for a single step right by (or any fixed distance away from) the origin. There are of course also many states that are almost surely transient, such as ones with non-monotonic domain walls or more generally ones that, restricted to some Q_L, do not satisfy the geometric conditions, such as those in the proof of Lemma 4.3, that prevent non-monotonic domain walls from forming.*

Remark 5.3. *The proof of Theorem 4 makes it clear that $\mathcal{W}$ must almost surely contain non-constant absorbing states both with horizontal and with vertical flat domain walls. Similarly, the single step domain walls mentioned in the previous remark will be both horizontal and vertical (and with the steps at all possible distances from the origin). It is a natural conjecture, discussed in Section 1, that almost surely all recurrent states, besides the two constant states, have only a single (monotonic) doubly-infinite domain wall. Two possibilities as to the exact class of recurrent states are discussed at the end of Section 1. We have not been able to show that almost surely some state with a single doubly-infinite domain wall is in fact recurrent. However, by using arguments like those in the proof of Theorem 4, one can show that there are almost surely recurrent (absorbing) states with a flat domain wall next to the origin and no other domain walls in a half-space.*

Acknowledgments. The research reported here was supported in part by the U.S. NSF under grants DMS-98-02310 (F.C.) and DMS-98-03267 (C.M.N.) and by the Italian CNR (E.D.S.). E.D.S. thanks the Courant Institute for its kind hospitality.

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